Operator Theory
Advances and Applications
Vol. 68

Editor
I. Gohberg

Time-Varying Discrete Linear Systems

Input-Output Operators.
Riccati Equations.
Disturbance Attenuation.

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Springer Basel AG

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A CIP catalogue record for this book is available from the Library of Congress, Washington D.C., USA

Deutsche Bibliothek Cataloging-in-Publication Data
Halanay, Aristide:
Time-varying discrete linear systems : input-output operators ;
Riccati equations ; disturbance attenuation / Aristide Halanay ;
Vlad Ionescu. – Basel ; Boston ; Berlin : Birkhäuser, 1994
(Operator theory ; Vol. 68)
ISBN 978-3-0348-9651-1 ISBN 978-3-0348-8499-0 (eBook)
DOI 10.1007/978-3-0348-8499-0
NE: Ionescu, Vlad:; GT

Originally published by Birkhäuser Verlag in 1994
Softcover reprint of the hardcover 1st edition 1994
Printed on acid-free paper produced from chlorine-free pulp
Cover design: Heinz Hiltbrunner, Basel
ISBN 978-3-0348-9651-1

9 8 7 6 5 4 3 2 1

To our wives, Maria and Adriana,
for their love, patience and support.

Preface

The present monograph has emerged from an attempt to develop a time-variant discrete compensation theory for achieving both stabilization and disturbance attenuation. After a short period of investigation it became clear to us that a systematic and coherent treatment of various subjects which are specific for time-variant discrete systems is needed. This forced us to confront several different topics such as exponential dichotomy; input-output operators between $l^2(\mathbb{Z})$ spaces; nodes, as the time-variant discrete counterpart of the ones studied by Bart, Gohberg and Kaashoek (see [5]) for the continuous case; Hankel and Toeplitz operators, and Liapunov and Riccati equations. To our knowledge such a treatment has never appeared in book form and we are convinced it will be useful to the reader in order to encourage the development of his own theoretical and practical work. In spite of the lack of monographs devoted to this subject, many publications do exist. Thus we have often found ourselves in something of a dilemma: on the one hand many facts should be known and on the other hand it is nearly impossible to give adequate reference to all. We must therefore apologize that our comments on references, made at the end of each chapter, are far from being complete. Moreover, we also rediscovered some results established a decade or more ago, such as those concerning stability via solutions to Liapunov equations or the ones related to exponential dichotomy deeply investigated by Ben-Artzi and Gohberg (see [7], [8], [9]). Thus we can not exclude the possibility that other results presented in the book for which we have no specific references were already known for some time. We feel that this situation argues all the more forcefully for writing a monograph on the subject.
At this point we wish to acknowledge the sources that influenced us and oriented our investigations. These were the theory of nodes due to Bart, Gohberg and Kaashoek [5]; the state-space approach to H^∞-control of Doyle, Glover, Khargonekar and Francis [18]; and the results of Popov and Yakubovich concerning the so-called "positivity theory" (see [55]).
The present form of the book is in fact the result of several revisions that were succesively performed on the initial version of the manuscript. The first two chapters were, for example, drastically modified. These modifications concern the structure of the material, examples and various new facts inspired by the recent volume edited by Gohberg, *Time-variant Systems and Interpolation* OT 56, *Operator Theory: Advances and Applications*, Birkhäuser, 1992. The third and fourth chapters also underwent radical changes before the present form was achieved. At present we believe that these chapters offer a new sight on the Riccati theory and disturbance attenuation problem as well. We are also conscious that our monograph is not one on operator theory, but that there are many operator-theoretical aspects disseminated in the text and a lot of facts may be more deeply imbedded in an operator framework. We are convinced that the well known interplay between operator theory and control system theory which is very transparent in transfer matrix terms must have also a state-space counterpart for which the time-varying case is of the greatest relevance.

We would like to warmly thank Professor Israel Gohberg for stimulating us to write this book and for publishing it in the series on *Operator Theory: Advances and Applications.*
We thank also Assistant Professor Mihai Tache for his dedication and skill in processing the text.
Finally, we are indebted to the Birkhäuser publishing staff for friendly and helpful assistance.

Aristide HALANAY

Bucharest 1993

Vlad IONESCU

Notation

Z	the set of integers
R^n	real n-dimensional Euclidean space
N	the set of natural numbers
$\triangleq$	defined by as well as defines
$\forall$	for all
$\square$	end of proof, lemma, remark, etc.
$l^2(Z, \mathbf{U})$	the Hilbert space of square summable $\mathbf{U}$-valued functions defined on $\mathbf{U}$
$l^2([s, \infty), \mathbf{U})$	the Hilbert space of square summable $\mathbf{U}$-valued functions with support in $[s, \infty) \subset Z$
$l^2((-\infty, s-1], \mathbf{U})$	the Hilbert space of square summable $\mathbf{U}$-valued functions with support in $(-\infty, s-1] \subset Z$
$\Vert\ \Vert_{\mathbf{X}}$	norm in Hilbert space $\mathbf{X}$
$<,>_{\mathbf{X}}$	inner product in Hilbert space $\mathbf{X}$
$\Vert\ \Vert_2$	l^2-norm
$<,>$	l^2-inner product
A^*	adjoint of the operator A
A^{-1}	inverse of the operator A
$A = A^* \gg 0$	$\exists\, \delta > 0,\ <Ax, x>_{\mathbf{X}} \geq \delta \Vert x \Vert_{\mathbf{X}}^2\ \ \forall\ x \in \mathbf{X}$
$\lambda(A)$	the spectrum of the operator A
$\rho(A)$	the spectral radius of the operator A

Cross references will follow the rule: Lemma 1 means lemma 1 in the same section; Lemma 2.1 means lemma 1 in the section 2 of the same chapter; Lemma 3.2.1 means lemma 1 in the section 2 of the chapter 3. The same rule applies to formulae: (1) means formula (1) in the same section; (2.1) means formula (1) in the section 2 of the same chapter; (3.2.1) means formula (1) in the section 2 of the chapter 3.

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Chapter 0

General motivation

Discrete-time systems have proved to be a subject of major interest in many scientific areas promoting intensive research equally disseminated both in theory and practice. Such systems arise naturally in modeling various types of processes but they are of crucial importance in Control Systems Theory. They are included in the 1991 Mathematical Subject Classification as 93C55 and in the version of sampled-data systems as 93C57. As such, the topics do not require supplementary motivation. What we would like to discuss here is the choice of matters and the structure of the present monograph. The starting point lies in the H^∞-optimization problem where model uncertainties have for the first time been systematically accounted for in the design. Initial examples of H^∞-solutions consisted solely of small numerical examples used to illustrate the theory. Now H^∞-optimization techniques are used for solving real-world problems arising from the area of the most advanced technologies. It is why H^∞-control has been finally included in the 1991 Mathematics Subject Classification as 93B36. In fact during the past decade the H^∞-optimization problem seems to be one of the most exciting research areas in Systems and Control Theory, and progress accomplished in this direction has been quite spectacular in the way it combined a sophisticated mathematical theory with practical engineering design considerations. The fascinating interplay between an engineering approach and advanced mathematical topics, mostly from Operator Theory and Complex Analysis but equally from Differential Equations and Linear Algebra, was the key of such rapid development of the field. Moreover we can conclude that a characteristic feature of control science is that mathematical and engineering advances have been closely intertwined at every stage of the development.

The H^∞-control theory was focused first on the continuous time-invariant systems but later it has been extended to the discrete-time and time-variant systems. Extension of the theory to general time-variant discrete systems is well motivated by the fact that when sampling a periodic continuous-time system one gets a discrete system with almost periodic coefficients.

In the second half of 1990 we started to study the *suboptimal* solution of the so-called *disturbance attenuation problem* which consists in finding a controller for a given *time-varying discrete* system such that closed-loop stability and regulated output attenuation with prescribed tolerance are simultaneously achieved. When the problem was completely solved (in terms of necessary and sufficient conditions) we discovered that it required a lot of specific results concerning discrete-time systems. Such results may be seen as being partitioned in two categories. The first category includes those results that we considered to be new such as a general Riccati theory for game-theoretic situations, developed in the perspective of the Popov-Yakubovich viewpoint. The second category, a rich one, consists of partially known results, or those which could be obtained by a specialist when necessary, but which never have been collected in a systematic way. The above considerations led us to write the present monograph. Let us remark that as we were stimulated to investigate specific aspects of time-variant systems starting from the disturbance attenuation problem,

Ball, Gohberg and Kaashoek developed a similar study motivated by the time-varying Nevanlinna-Pick interpolation theory (see [4]).
Let us be now more specific in order to have an idea concerning the topics we shall consider.
Let $\mathbf{X}$, $\mathbf{U}_i$, $\mathbf{Y}_i$, $i = 1,2$ be Hilbert spaces and let $A = (A_k)_{k \in \mathbf{Z}}$, $B_i = (B_{i,k})_{k \in \mathbf{Z}}$, $C_i = (C_{i,k})_{k \in \mathbf{Z}}$, $i = 1,2$ and $D_{ij} = (D_{ij,k})_{k \in \mathbf{Z}}$, $i,j = 1,2$ be bounded operator sequences i.e. $\sup_{k \in \mathbf{Z}} \left\{ \| A_k \| + \sum_{i=1}^{2} \| B_{i,k} \| + \sum_{i=1}^{2} \| C_{i,k} \| + \sum_{i,j=1}^{2} \| D_{ij,k} \| \right\} < \infty$ where $A_k : \mathbf{X} \to \mathbf{X}$, $B_{i,k} : \mathbf{U}_i \to \mathbf{X}$, $C_{i,k} : \mathbf{X} \to \mathbf{Y}_i$, $i = 1,2$ and $D_{ij,k} : \mathbf{U}_j \to \mathbf{Y}_i$, $i,j = 1,2$. Here we assume $D_{22} = 0$.
If $x = (x_k)_{k \in \mathbf{Z}}$ is any $\mathbf{X}$-valued sequence, let σ be the unit shift that is $(\sigma x)_k = x_{k+1}$. Write also Ax for the sequence $(A_k x_k)_{k \in \mathbf{Z}}$ i.e. consider A as a multiplication operator or, equivalently, as having a diagonal matrix representation where the diagonal entries equal A_i. With these in mind consider the linear discrete-time systems

$$\begin{aligned} \sigma x &= A x + B_1 u_1 + B_2 u_2 \\ y_1 &= C_1 x + D_{11} u_1 + D_{12} u_2 \\ y_2 &= C_2 x + D_{21} u_1 \end{aligned} \tag{1}$$

where $x = (x_k)_{k \in \mathbf{Z}}$, $u_i = (u_{i,k})_{k \in \mathbf{Z}}$, $y_i = (y_{i,k})_{k \in \mathbf{Z}}$, $i = 1,2$, with $x_k \in \mathbf{X}$, $(u_{1,k}, u_{2,k}) \in \mathbf{U}_1 \times \mathbf{U}_2$ and $(y_{1,k}, y_{2,k}) \in \mathbf{Y}_1 \times \mathbf{Y}_2$ are the *state*, the *exogenous input*, the *control input*, the *regulated output* and the *measured output* evolutions, respectively. The *disturbance attenuation problem* consists in finding a *controller*, i.e. a system

$$\begin{aligned} \sigma x_c &= A_c x_c + B_c y_2 \\ u_2 &= C_c x_c + D_c y_2 \end{aligned} \tag{2}$$

activated by the measured output y_2 and providing the control input u_2 such that the resultant closed loop system

$$\begin{aligned} \sigma x_R &= A_R x_R + B_R u_1 \\ y_1 &= C_R x_R + D_R u_1 \end{aligned} \tag{3}$$

does satisfy simultaneously the following two conditions

1. A_R defines an *exponentially stable evolution* i.e. $\| A_{R,i-1} A_{R,i-2} \cdots A_{R,j} \| \le \rho q^{i-j}$ for $\rho \ge 1$, $0 < q < 1$ and $\forall\, i > j$.

2. Once condition **1.** satisfied, system (3) defines a linear bounded input-output operator $T_{y_1 u_1} : l^2(Z, \mathbf{U}) \to l^2(Z, \mathbf{Y}_1)$ which must be such that it γ-attenuates the exogenous inputs that is $\| T_{y_1 u_1} \| < \gamma$, where γ is an apriori given tolerance.

Notice also that A_c, B_c, C_c, D_c are of the same operator nature as the coefficients of (1) and $x_c = (x_{c,k})_{k \in \mathbf{Z}}$, $x_{c,k} \in \mathbf{X}_c$ is the controller state evolution.
Let us explain a little more the origin and the relevance of the above stated problem which, in the time-invariant case, coincides with the well known H^∞-optimization problem (the suboptimal version). At a first inspection conditions **1.** and **2.** arise as standard requirements

imposed to a control system: a) closed loop stability and b) to keep y_1 "small", i.e. to achieve the attenuation condition $\| y_1 \|_2 < \gamma \| u_1 \|_2$ or, equivalently, $\| T_{y_1 u_1} \| < \gamma$. Notice that here y_1 must be seen as the classical *tracking error*. If y_1 is augmented, that is *more internal signals are considered as regulated outputs*, then by achieving the above mentioned attenuation condition the resultant closed loop configuration will be endowed with new remarkable properties. Such properties concern the so-called *robust stability*. The notion of robustness can be described as follows. Assume that the (operator) coefficients of the generalized system (1), i.e. A, B_i, C_i, D_{ij} , $i, j = 1, 2$ are perturbed. Thus we may consider (1) as belonging to a class **F** and, due to these perturbations, the system (1) ranges the class **F**. Usually such perturbations are viewed as *model uncertainties*. Consider also a characteristic of the closed loop system (3), for instance internal stability. We shall say that the controller (2) is *robust* with respect to this characteristic if this characteristic, i.e. internal stability, holds for every system in **F**. In order to argue in a deeper way the above mentioned robustness property consider first the so-called small gain theorem

Theorem 1 *(Small Gain). Let*

$$\begin{aligned} \sigma \tilde{x}_i &= \tilde{A}_i \tilde{x}_i + \tilde{B}_i \tilde{u}_i \\ \tilde{y}_i &= \tilde{C}_i \tilde{x}_i + \tilde{D}_i \tilde{u}_i \end{aligned} \qquad i = 1, 2 \tag{4}$$

be two internally stable systems defining the linear bounded input-output operators $\tilde{T}_i : l^2(\mathbf{Z}, \tilde{\mathbf{U}}_i) \to l^2(\mathbf{Z}, \tilde{\mathbf{Y}}_i)$, $i = 1, 2$. *Assume that the two systems are feedback compatible that is* $\tilde{\mathbf{U}}_2 = \tilde{\mathbf{Y}}_1$, $\tilde{\mathbf{Y}}_2 = \tilde{\mathbf{U}}_1$ *and* $(I - \tilde{D}_1 \tilde{D}_2)^{-1}$ *is well defined and bounded. If, for a given* $\gamma > 0$, $\| \tilde{T}_1 \| \le \gamma^{-1}$ *and* $\| \tilde{T}_2 \| < \gamma$ *then the resultant closed loop system* $\sigma \tilde{x}_R = \tilde{A}_R \tilde{x}_R$, $\tilde{x}_R = (\tilde{x}_1, \tilde{x}_2)$, *i.e. that system obtained by making* $\tilde{u}_1 = \tilde{y}_2$ *and* $\tilde{u}_2 = \tilde{y}_1$ *is internally stable* ($\tilde{A}_R$ *defines an exponentially stable evolution).* □

Remark 2. Theorem 1 asserts that if the first system (4) $(\tilde{A}_1, \tilde{B}_1, \tilde{C}_1, \tilde{D}_1)$ ranges the class **F** characterized by $\| \tilde{T}_1 \| \le \gamma^{-1}$ then the second system $(\tilde{A}_2, \tilde{B}_2, \tilde{C}_2, \tilde{D}_2)$ stabilizes the whole class **F**. □

In order to illustrate how robust stability is achieved let us connect together the disturbance attenuation problem and the small gain theorem. This will be done for a particular case of (1). To this end consider first

Lemma 3. *Assume that (1) reduces to*

$$\begin{aligned} \sigma x &= A x && + B_2 u_2 \\ y_1 &= && \quad\;\; u_2 \\ y_2 &= C_2 x + u_1 && \end{aligned} \tag{5}$$

Consider also the first system (4) assuming that $\tilde{\mathbf{U}}_1 = \mathbf{Y}_1$ *and* $\tilde{\mathbf{Y}}_1 = \mathbf{U}_1$. *Then the next two system operations lead to the same resultant system:*

1. Connect (2) to (5) and obtain (3). Connect then to (3) the first system (4) by making $u_1 = \tilde{y}_1$ *and* $\tilde{u}_1 = y_1$ *that is consider (3) as playing the role of the second system (4).*

2. Perturb additively the system

$$\begin{aligned} \sigma x &= A x + B_2 u_2 \\ y_2 &= C_2 x \end{aligned} \tag{6}$$

by the first system (4) and obtain

$$\begin{aligned} \sigma \begin{bmatrix} x \\ \tilde{x}_1 \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & \tilde{A}_1 \end{bmatrix} \begin{bmatrix} x \\ \tilde{x}_1 \end{bmatrix} + \begin{bmatrix} B_2 \\ \tilde{B}_1 \end{bmatrix} u_2 \\ y_2 &= [C_2 \quad \tilde{C}_1] \begin{bmatrix} x \\ \tilde{x}_1 \end{bmatrix} + \tilde{D}_1 u_2 \end{aligned} \tag{7}$$

and then connect to (7) the controller (2). □

The proof of Lemma 3 is obtained by performing simple computations.
Now we have

Theorem 4. *Let* $\gamma > 0$ *and assume that both* A *and* $\tilde{A}_1$ *in (7) define exponentially stable evolutions. If (2) is a solution to the disturbance attenuation problem formulated for (5), then (3) stabilizes (7) for all systems* $(\tilde{A}_1, \tilde{B}_1, \tilde{C}_1, \tilde{D}_1)$ *for which* $\| \tilde{T}_1 \| \leq \gamma^{-1}$*, that is (2) robustly stabilizes (6).*

Before proving the above stated theorem let us remark that by perturbing additively the system (6), it ranges the class $\mathbf{F} = \left\{ T_2 + \tilde{T}_1 \mid \| \tilde{T}_1 \| \leq \gamma^{-1} \right\}$ where T_2 is the input-output operator of (6).

Proof of Theorem 4. Apply Lemma 3 in conjunction with small gain Theorem 1. □

Thus we conclude that *robust stabilization of a given system* (see (6)) *reduces to solving the disturbance attenuation problem for an adequately generalized system* (see (5)). Therefore the disturbance attenuation problem plays a central role in the robustness theory.

Let us return to Theorem 1. In the continuous time-invariant case the proof of this theorem is a simple exercise in applying the Nyquist criterion which in fact is an engineering version of the variation of the argument formula. In our case such a treatement fails. In order to prove Theorem 1 we had to prove a more powerful result which is intimately related to the Popov positivity theory. Such result is stated as follows.

Theorem 5. *Let* $T : l^2(\mathbb{Z}, \mathbf{U}) \to l^2(\mathbb{Z}, \mathbf{Y})$ *be the input-output operator defined by the exponentially stable system* $\sigma x = A x + B u$, $y = C x + D u$. *Then, for a given* $\gamma > 0$, $\| T \| < \gamma$ *iff the following Kalman-Szegö-Popov-Yakubovich system in the so called positivity form*

$$\begin{aligned} \gamma^2 I - D^* D + B^* \sigma X B &= V^* V \\ -C^* D + A^* \sigma X B &= W^* V \\ -C^* C + A^* \sigma X A - X &= W^* W \end{aligned} \tag{8}$$

has a stabilizing solution (X, V, W), i.e. there exist bounded operator sequences $X = X^* = (X_k)_{k \in Z}$, $V = (V_k)_{k \in Z}$, $W = (W_k)_{k \in Z}$ for which (8) holds, V^{-1} is well defined and bounded and $A - B V^{-1} W$ defines an exponentially stable evolution. Moreover $X \leq 0$. □

Based on this result we have immediately

Proof of Theorem 1 (sketch).

Assume without loss of generality that $\| \tilde{T}_1 \| < \gamma^{-1}$. Then by applying adequately Theorem 5 to both systems (4) we may write two Kalman-Szegö-Popov-Yakubovich systems of type (8) with the stabilizing solutions (X_1, V_1, W_1) and (X_2, V_2, W_2), respectively, and where $\tilde{X}_1 \leq 0$ and $\tilde{X}_2 \leq 0$. Let

$$\tilde{X}_R \triangleq \begin{bmatrix} -\gamma^2 \tilde{X}_1 & 0 \\ 0 & -\tilde{X}_2 \end{bmatrix} \geq 0$$

Then simple computations lead to the Liapunov equation $\tilde{X}_R = \tilde{A}_R^* \sigma \tilde{X}_R \tilde{A}_R + \tilde{C}_R^* \tilde{C}_R$ where $\tilde{A}_R$ is associated to the resultant closed-loop system obtained by connecting together the systems (4) and $\tilde{C}_R$ is adequately defined. It is shown that the pair $(\tilde{C}_R, \tilde{A}_R)$ is detectable. This fact combined with $\tilde{X}_R \geq 0$ which satisfies the above Liapunov equation provides the exponentially stable evolution defined by $\tilde{A}_R$ and the proof ends. □

Let us now be a little more involved in the disturbance attenuation problem, for which we need firstly

Definition 6. Call $\Sigma = (A, B; M)$, where

$$M = \begin{bmatrix} Q & L \\ L^* & R \end{bmatrix} = M^*$$

and A defines an exponentially stable evolution, a Popov triplet. Here $A = (A_k)_{k \in Z}$, $B = (B_k)_{k \in Z}$, $M = (M_k)_{k \in Z}$ are bounded operator sequences where $A_k : \mathbf{X} \to \mathbf{X}$, $B_k : \mathbf{U}_1 \times \mathbf{U}_2 \to \mathbf{X}$, $M_k : \mathbf{X} \times \mathbf{U}_1 \times \mathbf{U}_2 \to \mathbf{X} \times \mathbf{U}_1 \times \mathbf{U}_2$, and $\mathbf{X}$, $\mathbf{U}_1$, $\mathbf{U}_2$ are Hilbert spaces.

Associate to Σ:

1. The Popov index

$$J(k,\xi,u) \triangleq \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} Q & L \\ L^* & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle \tag{9}$$

defined for all $k \in Z$ and $(\xi,u) \in \mathbf{X} \times l^2([k, \infty), \mathbf{U}_1) \times l^2([k, \infty), \mathbf{U}_2)$ $([k, \infty) \subset Z)$ where x and u are linked by $\sigma x = A x + B u$, $x_k = \xi$.

2. The Kalman-Szegö-Popov-Yakubovich system in "J form"

$$\begin{aligned} R + B^* \sigma X B &= V^* J V \\ L + A^* \sigma X B &= W^* J V \\ Q + A^* \sigma X A - X &= W^* J W \end{aligned} \tag{10}$$

where

$$J = \begin{bmatrix} -I_1 & \\ & I_2 \end{bmatrix} \quad , \quad V = \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} \tag{11}$$

with I_i the identities in $l^2([k\,,\infty)\,,\mathbf{U}_i)$, $i = 1\,,2$ and V partitioned in accordance with J.

3. The triplet $(X\,,V\,,W)$ is called a stabilizing solution to (10) if it satisfies (10), $X = X^*$, V^{-1} is well defined and bounded and $A + B\,F$ defines, for $F = -V^{-1}\,W$, an exponentially stable evolution. □

Remark 7. Notice that the exponentially stable assumption made on A can be easily removed. We needed it in order to simplify the presentation. In fact we invoke here the so-called "feedback invariance". □

By simple computation we have

Proposition 8. *Let Σ be a Popov triplet and assume that the associated Kalman-Szegö-Popov-Yakubovich system in J form has a stabilizing solution $(X\,,V\,,W)$. Then the Popov index can be expressed as*

$$\mathbf{J}(k,\xi,u) = -\gamma^2\|\,\tilde{u}_1\,\|_2^2 + \|\,\tilde{y}_1\,\|_2 \tag{12}$$

for all $(k,\xi) \in Z \times \mathbf{X}$ and all $u = (u_1\,,u_2) \in l^2([k\,,\infty)\,,\mathbf{U}_1) \times l^2([k\,,\infty)\,,\mathbf{U}_2)$ and where

$$\gamma\,\tilde{u}_1 \triangleq V_{11}u_1 + W_1x \tag{13}$$

$$\tilde{y}_1 \triangleq V_{21}u_1 + V_{22}u_2 + W_2x \tag{14}$$

with x and u linked by $\sigma x = A\,x + B\,u = A\,x + B_1\,u_1 + B_2\,u_2\,,\ x_k = \xi$ and $W^* = [W_1^* \quad W_2^*]$ partitioned conformaly with V in (11). □

Assume now that A in (1) defines an exponentially stable evolution. As we mentioned and as we shall argue a little more at the end of this chapter such assumption does not restrict the generality of the problem.
Let

$$B \triangleq \begin{bmatrix} B_1 & B_2 \end{bmatrix} \;,\; Q \triangleq C_1^*C_1 \;,\; L \triangleq C_1^*\begin{bmatrix} D_{11} & D_{12} \end{bmatrix}$$

$$R \triangleq \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix}\begin{bmatrix} D_{11} & D_{12} \end{bmatrix} - \begin{bmatrix} \gamma^2 I_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{15}$$

The Popov triplet Σ constructed with data defined by (15) will be called the *Popov triplet associated to the generalized system (1).*
We have at once

Proposition 9. *The Popov index corresponding to the Popov triplet Σ associated to (1) can be expressed as*

$$\mathbf{J}(k,\xi,u) = -\gamma^2\|\,u_1\,\|_2^2 + \|\,y_1\,\|_2^2 \tag{16}$$

□